

10 Recurrence Relations

10.1 First Order Recurrence Relations

In this chapter we look at recursively defined sequences (a_0, a_1, a_2, \dots) of numbers. A *recurrence relation* is an equation, or rule, that defines the n^{th} number a_n in terms of earlier numbers.

Example 10.1. The equation $a_n = 5a_{n-1}$ is a recurrence relation that is satisfied by many sequences of numbers: $(0, 0, 0, \dots)$ is one. $(1, 5, 25, \dots)$ and $(2, 10, 50, \dots)$ are others.

Observe that upon choosing a first term a_0 , (indeed any term in the example above,) the recurrence equation defined the whole sequence.

Example 10.2. Setting $a_0 = 3$ and $a_n = 5a_{n-1}$ for $n \geq 1$ defines the sequence $(a_0, a_1, a_2, \dots) = (3, 15, 75, \dots)$. One can show by induction that a_n has an explicit representation as $a_n = 3 \cdot 5^n$.

In general, the recurrence relation

$$a_0 = A, \quad a_{n+1} = d \cdot a_n \text{ for all } n \geq 0$$

defines the sequence with explicit formula $a_n = Ad^n$ for all $n \geq 0$. Such an explicit formula is called the solution of a recurrence relation.

Problem 10.1. You invest \$1000 in the bank at .5% interest per month. How much is it worth after 2 years?

Sol:

The value v_n in dollars after n months is determined by the initial value $v_0 = 1000$, and the recurrence relation $v_{n+1} = v_n \cdot 1.005$. So it is $v_{24} = 1000(1.005)^{24}$ dollars.

The above recurrence relations are quite simple. They are called *homogeneous first order linear recurrence relations with constant coefficients*, and we have solved them all.

A recurrence relation that can be written in the form

$$c_0 a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + d$$

is called a k^{th} order linear recurrence relation. If $d = 0$, it is called *homogeneous* and if the c_i are constant, it is said to have *constant coefficients*.

In later sections we solve all homogeneous second order linear recurrences with constant coefficients, and then some non-homogeneous recurrences. These take more theory. To finish off this section we look at some special cases that we can do without much more theory.

Problem 10.2. You invest \$100 per month in a bank at an interest rate of .5% per month. How much money do you have after one year?

Sol:

This is the recurrence $a_n = 100 + (1.005)a_{n-1}$ with $a_0 = 100$ we solve this more generally in the next example.

Example 10.3. Solve the (non-homogeneous) recurrence relation defined by $a_0 = 0$ and $a_{n+1} = da_n + c$.

Sol:

Writing out the first couple terms we have that $a_1 = c$, $a_2 = c(1+d)$, $a_3 = c(1+d+d^2)$. We can show by induction that this pattern continues and has the solution $a_n = c(1+d+d^2+\dots+d^{n-1}) = c\frac{d^n-1}{d-1}$.

Example 10.4. Solve the (non-linear) recurrence relation defined by $a_0 = 5$, and $a_{n+1}^3 = 2a_n^3$.

Sol:

It isn't linear in a_n , but making a change of variable $b_n = a_n^3$, the recurrence $b_0 = 5$ and $b_{n+1} = 2b_n$ is, and has solution $b_n = 5 \cdot 2^n$. So $a_n = (5 \cdot 2^n)^{1/3}$ is the solution to our original relation.

10.2 Second order linear homogeneous recurrence relations with constant coefficients

Second order linear homogeneous recurrence relations can be written as

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} = 0.$$

To solve this, we will guess a solution of the form $a_n = cr^n$ for non-zero c and r , and plug it in to the recurrence relation.

Now $c_0(cr^n) + c_1(cr^{n-1}) + c_2(cr^{n-2}) = 0$ if and only if $c_0r^2 + c_1r + c_2 = 0$. This is only true if r is a root of the *characteristic equation* $c_0x^2 + c_1x + c_2 = 0$. Let's see how this plays out in an example.

Example 10.5. (Distinct Real Roots) Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 5$.

Sol:

1) Rewriting the recurrence relation as $a_n - 5a_{n-1} + 6 = 0$, the characteristic equation is $x^2 - 5x + 6 = 0$.

2) The characteristic equation has *characteristic roots* $r = 2, 3$.

3) Each of these is a solution to the recurrence relation. As it is linear, any linear combination $A(2^n) + B(3^n)$ is also a solution, so we have the *general solution*

$$a_n = A(2^n) + B(3^n)$$

(for constants A and B) to the recurrence $a_n = 5a_{n-1} - 6a_{n-2}$.

4) Using $a_0 = 1$ and $a_1 = 5$, we get a *specific solution*. From the first *initial condition* we get, $1 = a_0 = A(2^0) + B(3^0) = A + B$. From the second, we get $5 = a_1 = A(2) + B(3) = 2A + 3B$. Solving these, we get $B = 3$ and $A = -2$. So the specific solution to our recurrence relation is

$$a_n = (-2)2^n + (3)3^n = 3^{n+1} - 2^{n+1}.$$

Remark 10.6. It can be shown, using linear algebra, that the the general solution given in part 3) describes all possible solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$. Indeed, the recurrence can be rewritten $a_n - 5a_{n-1} - 6a_{n-2} = 0$. As a system of equations in the variables a_n, a_{n-1} , and a_{n-2} the solution space has dimension 2. We won't prove this, we will just believe it.

This lays out a general process for solving linear homogeneous recurrence relations with constant coefficients.

1. Find the characteristic equation.
2. Find the characteristic roots.
3. Find the general solution.
4. Use the initial conditions to find the specific solution.

Try this process to solve the following.

Problem 10.3. Solve the following (shifted fibonacci) recurrence relation: $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, and $f_1 = 1$. (See Example 10.10 of the text book. You should get that $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$).

This process works for higher order relations than 2^{nd} order, however we may have trouble finding the characteristic roots for higher order equations. We will mostly only consider 2^{nd} order equations. The previous example was the easiest case. In general, our characteristic roots may not be real. This is not a difficult complication. The more difficult complication is when they are real, but not distinct. (If they aren't real, they will be distinct.)

Example 10.7. (Complex Roots - Necessarily Distinct) Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$ with initial conditions $a_0 = a_1 = 1$.

Sol:

There characteristic equation is $x^2 - 2x + 2 = 0$, which has roots $x = 1 \pm \sqrt{-1} = 1 \pm i$. So our general solution is

$$a_n = A(1 + i)^n + B(1 - i)^n.$$

We now find the specific solution. For one, we have $1 = a_0 = A + B$, so $B = 1 - A$. From the second initial condition, we have

$$1 = a_1 = A(1+i) + B(1-i) = A(1+i) + (1-A)(1-i) = 2Ai + 1 - i.$$

Solving give $A = B = 1/2$, so we get the specific solution

$$a_n = 1/2(1+i)^n + 1/2(1-i)^n.$$

Example 10.8. (Repeated real roots) Solve $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 3$ and $a_1 = 10$.

Sol:

The characteristic equation $x^2 - 6x + 9 = (x - 3)^2 = 0$ has double root 3. In this case, we have a solution $a_n = A3^n$, for all A , but this is not all of the solutions. Observe that $n3^n$ is also a solution:

$$n3^n = 6(n-1)3^{n-1} - 9(n-2)3^{n-2}.$$

Again linearity gives that

$$a_n = A3^n + Bn3^n$$

is a solution for all A and B . This is our general solution.

Using the initial conditions, we can solve to get the specific solution $a_n = 3^{n+1} + n3^{n-1}$.

Problem 10.4. Solve $a_n = -2a_{n-1} - a_{n-2}$ where $a_0 = 2$ and $a_1 = 5$.

Sol:

You should find that the characteristic equation has a double root -1 , and get a specific solution $(2 - 7n)(-1)^n$.

As long as we can find the characteristic roots, we should be able to solve higher order homogeneous linear equations with constant co-efficients.

Example 10.9. (Third Order) Solve $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$, where $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.

Sol:

The characteristic equation $2x^3 - x^2 - 2x + 1 = 0$ has roots $x = \pm 1, 1/2$. So the general solution is $a_n = A(1)^n + B(-1)^n + C(1/2)^n$. Now we find the particular solution. From the first initial condition, $0 = a_0 = A + B + C$, from the second $1 = a_1 = A - B + C/2$, from the third we have $2 = a_2 = A + B + C/4$. This is three equations in three unknowns, so we can solve it to get $A = 5/2$, $B = 1/6$, and $-8/3$. So $a_n = \frac{5}{2} + (\frac{-1}{6})^n - \frac{8}{3}(\frac{1}{2})^n$.

Now let's see a bit how we can use these recurrence relations.

Problem 10.5. How many binary strings (strings of '0's and '1's) of n bits have no two consecutive zeros?

Sol:

Let a_n be the number of such strings. Clearly $a_0 = 0$, $a_1 = 2$, and $a_2 = 3$. An n bit string begins (1, and ends with any $n - 1$ bit string, or begins with (0, 1, and ends with any $n - 2$ bit string. So $a_n = a_{n-1} + a_{n-2}$. We can solve this.

This is almost the same problem.

Problem 10.6. I have buttons of four different colours. How many ways can we arrange n of them so that there are no two consecutive blue ones.

Sol:

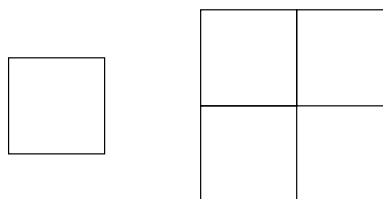
You should get a recurrence relation $a_n = 3a_{n-1} + 3a_{n-2}$.

Problem 10.7. How many compositions of the integer n use only the digits 1 and 2?

Sol:

We add a 1 to the end of any composition of $n - 1$ and a 2 to the end of any composition of $n - 2$. So this is described by the recurrence relation $a_n = a_{n-1} + a_{n-2}$. The initial conditions are $a_1 = 1$ and $a_2 = 1$. Fibonacci fabulous!

Problem 10.8. How many ways can you tile a $2 \times n$ checkerboard with the



tiles of the type seen here.

Sol:

This is Example 10.19 from the text.

10.3 The Nonhomogeneous Recurrence Relation

A linear recurrence relation, written as

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} = f(n)$$

is called *nonhomogeneous* if $f(n)$ is not identically 0.

We will look at the first and second order cases. Again, one can show by linear algebra that to find the general solution to such a relation, one only needs to find the general solution $a_n^{(h)}$ to a *corresponding homogeneous recurrence relation*, and then shift it by one *particular solution* $a_n^{(p)}$.

We start with an example.

Example 10.10. Solve $a_n = 2a_{n-1} + 7 \cdot 5^n$ for $n \geq 1$ and $a_0 = 4$.

Sol:

First we find the general solution to the corresponding homogeneous recurrence $a_n - 2a_{n-1} = 0$. This has characteristic equation $x - 2 = 0$ which has root $x = 2$ so we have the general solution $a_n^{(h)} = A \cdot 2^n$.

Now we want a particular solution to $a_n - 2a_{n-1} = 7 \cdot 5^n$. Lets guess that there is one of the form $c \cdot 5^n$. (We talk about why after this example.) Plugging into the recurrence relation, we get

$$c \cdot 5^n - 2c \cdot 5^{n-1} = 7 \cdot 5^n,$$

which is true if and only if $c \cdot 5 - 2c = 35$, or $c = 35/3$. So $a_n^{(p)} = (35/3) \cdot 5^n$ is a solution to the recurrence.

The general solution to the recurrence is thus

$$a_n = a_n^{(h)} + a_n^{(p)} = A \cdot 2^n + (35/3) \cdot 5^n.$$

Using the initial conditions we now find the specific solution. We have $4 = a_0 = A(2^0) + (35/3)5^0 = A + 35/3$, which gives $A = -23/3$. The specific solution is thus

$$a_n = (35/3)5^n - (23/3)2^n = (1/3)(35 \cdot 5^n - 23 \cdot 2^n).$$

For $f(n) = 7 \cdot 5^n$ we guessed a solution of the form $c \cdot 5^n$. The solution that we guess is suggested by the form of $f(n)$:

When $f(n)$ is	$a_n^{(p)}$ is of the form
c	A
cn	$An + B$
cn^2	$An^2 + Bn + C$
cr^n	Ar^n
cn^2r^n	$r^n(An^2 + Bn + C)$

From the above, you can interpolate appropriate guesses for various other special cases of $f(n)$. You will be responsible only for solving nonhomogeneous equations with $f(n)$ from the above table.

Problem 10.9. Solve $a_n = -6a_{n-1} - 9a_{n-2} + (-3)^n$ for $n \geq 2$, where $a_0 = 2$ and $a_1 = 3$.

Example 10.11. (The Towers of Hanoi) In the game of the Towers of Hanoi one has three pegs. A stack of disks of different radii are stacked on one peg with no disk on top of a larger disk. The game is to move the stack of disks to another peg. In a move, one can move one disk, placing it on an empty peg, or on a peg of smaller radius. How many moves can this be done in when there are n disks.

Sol:

Let a_n be the minimum number of moves one can complete the game in when there are n disks. Clearly $a_1 = 1$. To move n disks one must move the largest disk. When this is done, it must be moved from peg 1 to peg 3, so at this time all other disks must be stacked on peg 2. Up until this move, there must be at least a_{n-1} moves, and after the biggest disk is moved, we must move these other disks on top of it. This takes a_{n-1} moves. So $a_n = 2a_{n-1} + 1$. We have a recurrence relation which we can solve.

The corresponding homogeneous equation has characteristic polynomial $x-2 = 0$ so general solution $a_n^{(h)} = A \cdot 2^n$. The nonhomogeneous portion of the recurrence relation is 1 which suggests a particular solution of the form B . Plugging in, we get $B = 2B + 1$ which we solve for the particular solution $a_n^{(p)} = -1$. So the nonhomogeneous recurrence has general solution $a_n^{(h)} = A \cdot 2^n - 1$.

Using the initial condition that $1 = a_1 = A \cdot 2 - 1$, we get $A = 1$, so the specific solution is thus $a_n = 2^n - 1$.

The best one can do for the n disk game is $2^n - 1$ moves.