Discrete Math Final

Each question is worth 5 points.

1. You roll a fair die (dice) three times. What is the probablility that at least one of the following are true: (a) all rolls are even, or (b) the sum of the rolls is 10.

   Sol: (a) There are $3^3 = 27$ ways that all rolls are even. (b) The number of ways the rolls sum to 10 is the number of ways that the first two rolls are between 4 and 9, so are not $2, 3, 10, 11$, or 12. This is $36 - (1 + 2 + 3 + 2 + 1) = 27$ ways. The number of rolls that satisfy (a) and (b) is the number of ways that the first two rolls are even and sum to $4, 6$, or 8. This is $1, 2$ and 3 ways respectively, so 6 ways.

   The number of ways at least one of (a) and (b) is true is thus $27 + 27 - 6 = 48$. The probability is thus $48/6^3 = 2/9$.

2. Prove that for any integer $n \geq 24$, n can be written as a sum of '5's and '7's.

   Sol: The proof is by strong induction. Our base cases are the values $n = 24, \ldots, 28$ which we verify by $24 = 5 + 5 + 7 + 7$, $25 = 5 + 5 + 5 + 5 + 5$, $26 = 5 + 7 + 7 + 7$, $27 = 5 + 5 + 5 + 5 + 7$ and $28 = 7 + 7 + 7 + 7$. Now let $k > 28$ and assume that the statement is true for $n = k - 5$. That is, assume that $k - 5$ can be represented as a sum of '5's and '7's. Then $k = 5 + (k - 5)$, can also be represented as such sum. So the statement follows by induction.

3. For all integers a, b, c and d,

   (a) [1 pt] Define $a \mid b$.

   Sol: $a \mid b$ means there exists an integer x such that $ax = b$.

   (b) [2 pt] Prove $a \mid c \Rightarrow a \mid bc$

   Sol: Assume that $a \mid c$, so there is an integer x such that $c = ax$. So $bc = bax = a(xb)$. As xb is an integer, this gives that $a \mid bc$.

   (c) [2 pt] Prove that if $a \nmid bc$ then $a \nmid b$ and $a \nmid c$.

   Sol: We show that the contrapositive is true: that if $a \mid b$ or $a \mid c$, then $a \mid bc$. Indeed, if either $a \mid b$ or $a \mid c$, then $a \mid bc$ by part (b).

4. (a) [1pt] For positive integers a and b, define $\gcd(a, b)$.

   Sol: gcd(a,b) is the unique positive integer d that satisfies

   - $d \mid a$,
   - $d \mid b$, and
   - $c \mid a$ and $c \mid b$ imply that $c \mid d$

   (b) [4pt] Prove that if $d = a + bc$ then $\gcd(b, d) = \gcd(a, b)$.

Sol: We show that $\gcd(b,d)|\gcd(a,b)$ and $\gcd(a,b)|\gcd(b,d)$. As both are positive, it follows that they are equal.

For the first, observe that by definition, $\gcd(b,d)$ divides b and d. So it divides the linear combination $a = d - bc$. As it divids a and b it divides $\gcd(a,b)$ by the definition of $\gcd(a,b)$.

For the other statement, observe that $\gcd(a,b)$ divides a and b, and so divides $d = a + cb$. As it divides b and d, by definition of $\gcd(b,d)$, we have $\gcd(a,b)|\gcd(b,d)$, as needed.

Thus $\gcd(b,d) = \gcd(a,b)$.

5. For integers m,n with $mn = 2^4 3^4 5^2 11^2 37^{11}$, if $\text{lcm}(m,n) = 2^2 3^3 5^1 11^2 37^7$ then what are the following?

   (a) [2 pt] $\gcd(m,n)$

   Sol: We have $mn = \gcd(m,n)\text{lcm}(m,n)$, so $\gcd(m,n) = 2^2 3^1 5^1 37^4$.

   (b) [2 pt] $\gcd(m,mn)$

   Sol: Clearly m divides m and mn, so $m \mid \gcd(m,mn)$. But $\gcd(m,mn) \mid m$ by definition, so $\gcd(m,mn) = |m|$.

   (c) [1 pt] $mn \mod 3m$

   Sol: As $3 \mid \gcd(m,n)$, we have $3 \mid n$ so $3m \mid nm$. Thus $mn \mod 3m$ is 0.

6. Solve the following recurrence relation.

$$a_n = a_{n-1} + 6a_{n-2} + 5^n \text{ for } n \geq 2,$$

$a_0 = 0$, $a_1 = 1$.

Sol: The associated homogeneous characteristic equation is

$$x^2 - x + 6 = 0,$$

which has roots 3 and -2 (both of multiplicity one). So the general homogeneous solution is

$$a_n^{(h)} = A(3)^n + B(-2)^n.$$

Since the non-homogeneous part of the equation is 5^n, the particular solution of the non-homogeneous equation is of the form

$$a_n^{(p)} = C(5)^n.$$

Plugging this into the original equation yields that $C = 25/14$. Thus the general solution is of non-homogeneous

$$a_n = A(3)^n + B(-2)^n - (25/14)(5)^n.$$

Using initial conditions, we find that $B = (18/35)$ and $A = (-23/10)$. So

$$a_n = (-23/10)(3)^n + (18/35)(-2)^n + (1/14)(5)^{n+2}.$$

7. Find (but don't solve) a recurrence relation that describes the number of n digit sequences of the digits $(0, 1, 2, 3)$ in which there is never an odd digit anywhere to the right of an even digit.

   Sol: Let f_n be the number of such strings of length n. If the last (rightmost) digit is odd (that is 1 or 3), then all other digits are even, so there are 2^{n-1} to fill in the first $n-1$ digits. If the last digit is even (that is 0 or 2), then there are $fn - 1$ to fill in the first $n - 1$ digits. So $f_n = 2f_{n-1} + 2(2)^{n-1} = 2f_{n-1} + 2^n$ for $n \geq 2$, and $f_1 = 4$.

8. How many non-isomorphic spanning trees are there of $K_{2,3}$? (Give an argument to support your answer.)

   Sol: A spanning tree on 5 vertices has 4 edges. As each vertex on the bipartite set with two vertices must have an edge (because a tree is connected), they have respective degrees 1 and 3 or 2 and 2. Upto isomorphism, there is only one spanning tree satisfying each of these conditions. Thus there are 2.

9. Let G be a graph that contains a circuit which contains the vertex w. Prove that G contains a cycle that contains the vertex w.

   Sol: Let G be a graph that contains a circuit which contains the vertex w. Let $w = v_1, v_2, \ldots, v_r, v_1$ be the shortest circuit in G containing w. We claim that this circuit is indeed a cycle, for assume that it isn't. Then there exists some $1 \leq i < j \leq r$ such that $v_i = v_j$. But then $v_1, \ldots, v_i, v_{j+1}, \ldots, v_r, v_1$ is a shorter circuit containing w. This contradicts the fact that our orignal circuit was chosen to be the shortest. Thus the assumption that it wasn't a cycle is false.

10. If G is a graph on $n \geq 2$ vertices, and G is not connected, then show that $\overline{G}$ is connected.

    Sol: The case that $n = 2$ is trivial, so assume that $n \geq 3$. As G is not connected, there are vertices u, v in G such that G contains no uv-walk. In particular, uv is not an edge in G, so is in $\overline{G}$. For any other vertex w in G, there cannot be both a uw-path and a vw-path in G, otherwise there would be a uv-walk, which is a contradiction, so in particular at least one of uw and vw are not edges in G. So at least one of them is an edge in $\overline{G}$. Thus for any w and w' in $\overline{G}$, there is a ww'-walk in $\overline{G}$ containing u or v or both u and v. So $\overline{G}$ is connected.

    Alternately, use induction on n. Again the base case, $n = 2$ is trivial. So let G be a diconnected graph with $n \geq 3$ vertices, and assume the statement is proved for all graphs on fewer than n vertices. As G is disconnected, and has at least 3 vertices, there is some vertex v such that $G' = G \setminus \{v\}$ is disconnected. So $\overline{G'}$ is connected by the induction hypothesis. As G is disconnected, there is some $u \in G$ such that uv is not an edge in G; otherwise, any two vertices in G are connected by a path through v. Thus v is adjacent to u in $\overline{G}$, so $\overline{G}$ is connected. The statement follows by induction.