Practice Test Solutions

1. (a) How many permutations are there of the letters $a, b, c, d, e, f, g, h, i$, and j?

   Sol: 10!

   (b) How many start with the letter a?

   Sol: There are 9! ways to arrange the other 9 letters, so 9!

   (c) How many of the permutations from (a) start with a vowel?

   Sol: There are 3 ways to choose the first letter, and then 9! ways to choose the others. So there are $3 \cdot 9!$ ways in total.

   (d) How many of the permutations from (a) have no adjacent vowels?

   Sol: We have 7! ways to arrange the constanants, and $\binom{8}{3}$ ways to put the vowels in the spaces between them. Thus there are $7! \cdot \binom{8}{3}$ ways.

   (e) How many of the permutations from (a) have all vowels together?

   Sol: Gluing the vowels together there are 8 elements, so 8! ways to arrange them. There are 3! ways to glue the vowels together, so $8! \cdot 3!$ arrangements in total.

2. (a) How many ways can 3 couples be arranged around a circular table. (Arrangements that differ by a rotation, are the same.)

   Sol: Fixing the first person, there are 5! ways to arrange the rest. So 5!

   (b) In how many of these arrangements do the couples sit together.

   Sol: Glueing the couples together, the first couple can be assumed fixed, so there are 2! ways to arrange them. There are two ways to glue each couple together, so $2! \cdot 2^3$.

   (c) In how many of the arrangements in (a) are exactly one couple seated together?

   Sol: There are 3 ways to choose the couple that sits together, and two ways to sit them together. Once they are seated, we may assume they are the first and second people. There are 4 ways to choose the third person, 2 ways to chose the fourth, and then all others are already determined. Thus there are $3 \cdot 2 \cdot 4 \cdot 2 = 48$ ways in all.

3. At the You-Mix-It trail mix store you get a choice between 5 kinds of nuts, 8 kinds of dried fruits, and 3 kinds of sweets. How many ways can you mix your You-Mix-It trail mix if

   (a) You must mix in 7 different items.

   Sol: $\binom{16}{7}$.

   (b) Of your 7 items, 2 must be nuts and 2 must be sweets.

   Sol: 3 must be fruits, so $\binom{5}{2} \cdot \binom{8}{3} \cdot \binom{3}{2}$

   (c) Of the 7 items, exactly 1 is a sweet, and least 3 are nuts.

   Sol: Three cases: 3 nuts, 4 nuts, 5 nut: $\binom{5}{3}\binom{8}{3}(3) + \binom{5}{4}\binom{8}{2}(3) + \binom{5}{5}\binom{8}{1}(3)$

4. In a standard poker deck, (In each of 4 suits, there is one card of each of 13 ranks.) a poker hand consists of 5 unordered cards. How many ways can a player get a

   (a) Fullhouse? (Three cards of one rank and two of another.)

Sol: The rank of the three cards can be chosen 13 ways, and then the suits can be chosen in $\binom{4}{3} = 4$ ways. The rank of the two cards can be chosen in in 12 ways and the suits cards in $\binom{4}{2} = 6$ ways. Thus: $13 \cdot 4 \cdot 12 \cdot 6$.

(b) Two Pair and nothing higher? (Two cards of one rank, two of another, and one of a third rank.)

Sol: $13 \cdot 12 \cdot 11$ ways to pick the ranks, $\binom{4}{2} = 6$ ways to choose the two suits in each pair, and 4 ways to choose the suit of the third rank. Then divide by 2 because the order we chose the pairs in doesn't matter. Thus: $13 \cdot 12 \cdot \cdot 6^2 \cdot 2$.

5. (a) How many integer solutions are there to the following

$$x_1 + x_2 + x_3 + x_4 = 8, x_i \geq 0, i = 1, \ldots, 4.$$

Sol: Arrange three dividers and 8 dots: $\binom{8+3}{3} = \binom{11}{3}$.

(b) How many ways are there to distribute 20 indentical dimes among 4 children If the youngest must get at least one dime, the second youngest must get at least two dimes, the second oldest must get at least 3 dimes, and the oldest must get at least 4 dimes.

Sol: This leaves 10 dimes to distrubute among the 4 children with no restrictions, so can be done in $\binom{13}{3}$ ways.

(c) For which positive integer $n \geq 19$ will the equations

$$x_1 + x_2 + \cdots + x_{19} = n, \text{ and}$$

$$x_1 + x_2 + \cdots + x_{64} = n,$$

have the same number of **positive integer solutions**. (Note that this question is a lot harder if you allow solutions with zero.)

Sol: **(A bit hard.)** The two equations have $\binom{(n-19)+18}{18} = \binom{n-1}{18}$ and $\binom{(n-64)+63}{63} = \binom{n-1}{63}$ solutions respectively. These are equal when $18 + 63 = n - 1$. That is, when $n = 82$.

6. Give all truth value assignments for the primitive statements p, q and r that make the following compound statement true

$$(p \to r) \to [(p \to q) \wedge (q \to r)].$$

Sol: Making a truth table, you will find that all assigments except $p = 0, q = 1, r = 0$ give the statement a true value.

7. Determine if $\neg(p \wedge \neg q) \to \neg p$ is a tautology.

Sol: The truth assignments $p = 1$ and $q = 1$ yield a false for the statements, so it isn't a tautology. (You could find this with a truth table.)

8. (a) Two n-digit numbers (leading zeros allowed) are equivalent if one is a rearrangement of the other. (For example, 00123 and 10203 are equivalent.) How many 5 digit integers are not equivalent?

Sol: This is the number of ways of choosing five items of 10 different types: $\binom{10+4}{4}$.

(b) What if the digits $3, 5$, and 7 can occur at most once each?

Sol: There are 4 cases: 0, 1, 2, or all of the digits $3, 5, 7$ are used. Summing these cases up we have $1 \cdot \binom{7+4}{4} + 3 \cdot \binom{7+3}{3} + 3 \cdot \binom{7+2}{2} + 1 \cdot \binom{7+1}{1}$ ways.

9.

(a) Using laws of inference, establish the validity of the argument $[(p \to (q \to r)) \land (\neg q \to \neg p) \land p)] \to r$

Sol: We will use proof by contradiction

1)	$\neg r$	Extra assumption.
2)	$p \to (q \to r)$	Given.
3)	p	Given.
4)	$q \to r$	2,3, Modus ponens.
5)	$\neg q$	1,4, Modus tollens
6)	$\neg q \to \neg p$	Given.
7)	$\neg p$	5,6, Modus ponens.
8)	F_0	3,7.
9)	r	1,8, proof by contradiction.

(b) Show that the following argument is invalid.

$$\begin{array}{c} p \land \neg q \\ p \to (q \to r) \\ \hline \therefore \neg r \end{array}$$

Sol: When p and r are true and q is false, then all of the premises are true, while the conclusion is false. So the argument is invalid.

10. Show that

$$\neg \forall x \forall y [p(x,y) \to q(x)] \iff \exists x \exists y [p(x,y) \land \neg q(x)].$$

Sol:

$$\begin{array}{lll} & \neg \forall x \forall y [p(x,y) \to q(x)] & \\ \iff & \exists x \neg \forall y [p(x,y) \to q(x)] & \\ \iff & \exists x \exists y \neg [p(x,y) \to q(x)] & \\ \iff & \exists x \exists y \neq [\neg p(x,y) \lor q(x)] & \text{Defn of } \to. \\ \iff & \exists x \exists y [\neg\neg p(x,y) \land \neg q(x)] & \text{Demorgan.} \\ \iff & \exists x \exists y [p(x,y) \land \neg q(x)] & \text{Double negation.} \end{array}$$

11. Let $A = \{1, 2, \ldots, 20\}$. **This isn't in the scope of the test.**

(a) What is $|\{x \mid (x \in A) \land (x \text{ is prime})\}|$?

Sol: $|\{2, 3, 5, 7, 11, 13, 17, 19\}| = 8$

(b) Give the set $\{x \mid (x \in A) \land (5x \in A)\}$.

Sol: $\{1, 2, 3, 4\}$

(c) What is $|\mathscr{P}(A)|$?

Sol: 2^{20}

(d) What is $|\mathscr{P}(\mathscr{P}(A))|$?

Sol: $2^{2^{20}}$

(e) What is the cardinality of $\{B \mid (B \in \mathscr{P}(A)) \land (|B| = 4)\}$?

Sol: This is the number of 4 element subsets of a 20 element set, so is $\binom{20}{4}$