Discrete Math Test 1

Each question is worth 10 points, distrubited evenly among the parts.

1. (a) How many arrangements (permutations) are there of the letters of the word “REDRESSES”?

   Sol: There are three E’s, three S’s, two R’s, and one D, for a total of nine letters. Thus we have $9!/(3! \cdot 3! \cdot 2!)$ arrangements.

   (b) How many arrangements are there in which all of the E’s are together?

   Sol: Gluing the E’s together there are seven items: three S’s, two R’s one D, and one EEE. They can be arranged in $7!/(3! \cdot 2!)$ different ways.

   (c) How many arrangements contain the word “DRESS” in that order?

   Sol: Gluing the word “DRESS” together leaves two more E’s, one more R, and one more S. This is five items. We arrange them in $5!/2!$ ways.

   (d) How many of the arrangements in (a) have no consecutive E’s?

   Sol: There are $6!/(3! \cdot 2!)$ arrangements of the other letters, and $\binom{7}{3}$ ways to place the E’s in the 7 spaces between (and around) them. Thus there are $\binom{7}{3} \cdot 6!/(3! \cdot 2!)$ such arrangements.

2. (a) A fruit stand has apples, blueberries, coconuts, dates, and elderberries. How many ways can you choose a selection of 20 different fruits?

   Sol: We can distribute 5 types, (so 4 ’seperators’) among 20 spots in $\binom{20+4}{4}$ ways.

   (b) How many ways can you do this if you must have at least one of each kind?

   Sol: We distribute the 5 types among the 15 remaining spots in $\binom{15+4}{4}$ ways.

   (c) How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 \leq 40$ when $x_i \geq 1$ for $i = 1, \ldots, 4$? (Notice that it is “≤ 40”, not “$= 40$”.)

   Sol: First pre assign one ’point’ to each variable thus reducing the problem to the equivalent on of counting non-negative solutions of the inequality

   $$x_1 + x_2 + x_3 + x_4 \leq 36.$$

   Adding a ’slack’ term x_5 to the sum, we can make the inequality into an equality. There are then $\binom{36+4}{4}$ ways to distrubute the 36 remaining ’points’ among the five variables.

3. (a) On the euclidean plane how many distinct paths are there from $(0, 1)$ to $(5, 5)$ using only the following movements

   - $(x, y) \to (x + 1, y)$
   - $(x, y) \to (x, y + 1)$

   Sol: This is the number of ways to arrange four UP’s and five RIGHT’s, which is $9!/(5! \cdot 4!) = 126$ ways.

   (b) How many of these paths don’t use the move $(2, 3) \to (3, 3)$.

Sol: It's easier to count the paths that do. There are $4!/(2! \cdot 2!) = 6$ paths from $(0,1)$ to $(2,3)$, and $4!/(2! \cdot 2!) = 6$ paths from $(3,3)$ to $(5,5)$, so there are $6 \cdot 6 = 36$ paths that use the move $(2,3) \to (3,3)$. Thus there are $126 - 36 = 90$ paths that don't.

4. (a) Construct a truth table for the statement $[p \wedge (p \to q)]$.
Is the statement a tautology?

   Sol: It isn't a tautology.

   (b) Verify that $[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$ is a tautology. (This should be really quick.)

   Sol: By the law of inverse $s \wedge \neg s$ is tautology. Substituting the statement $[p \vee (q \wedge r)]$ for s gives that the desired statement is a tautology.

5. (a) Using laws of inference, establish the validity of the argument $[(p \to q) \wedge (r \to \neg q) \wedge r)] \to \neg p$

   Sol:

1)	$r \to \neg q$	Given.
2)	r	Given.
3)	$\neg q$	1,2, Modus ponens.
4)	$p \to q$	Given.
5)	$\neg p$	3,4, Modus tollens.

   (b) Show that the following argument is invalid.

$$\begin{array}{l} p \leftrightarrow q \\ q \to p \\ r \vee \neg s \\ \neg s \to q \\ \hline \therefore s \end{array}$$

   Sol: If q, r, and p are true and s is false, all of the premises are true, while the conclusion is false. So the argument is invalid.

6. Given the statements
$p(x)$: $(x-2)(x-5) = 0$
$r(x)$: $x < 0$

Determine the truth or validity of the following statements, where the universe for x and y is all integers. (Give explanations.)

   (a) $\forall x[p(x) \to \neg r(x)]$

   Sol: $p(x)$ is true only when $x = 2, 5$ and in both these cases $\neg r(x)$ is true. Thus the statement is valid.

   (b) $\exists x[p(x) \to r(x)]$

   Sol: This is valid because when x is say 1, $p(x)$ is false, and so the statement in brackets is true.

   (c) $\forall x \exists y[p(x) \to r(y)]$

   Sol: This is valid, because for any x we can take y to be -1, which makes $r(y)$ true, and so the statement in the brackets true.

   (d) $\exists x \forall y[p(x) \wedge (x + y = 1)]$

   Sol: This is invalid, because for any x there are many y for which $x + y = 1$ isn't true.