

# Discrete Math Midterm Solutions

1. Prove or disprove: If there is an isomorphism  $\phi : G \rightarrow H$  between graphs  $G$  and  $H$ , then there exist a bijection  $f : E(G) \rightarrow E(H)$  between their edge-sets.

## Solution

This is true. The bijection  $f : E(G) \rightarrow E(H)$  is defined for  $e = (u, v)$  in  $E(G)$  by

$$f((u, v)) = (\phi(u), \phi(v)).$$

It is an injection because  $\phi$  is, and is a surjection onto  $E(H)$  because  $\phi$  is an isomorphism.

2. (a) Draw the graph  $C_4$ .  
 (b) How many non-isomorphic subgraphs of  $C_4$  are there with 4 vertices?

## Solution

$C_4$  is a four cycle. It only has 4 edges.

On four edges: 1; on three edges: 1, on two edges: 2, on one edge: 1; on zero edges: 1. So 6 in total.

3. (a) Give an example of a **non-graphic** sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ , with  $1 \leq d_i \leq n - 1$  for all  $i$ , which contains an even number of odd integers.  
 (b) Decide if the sequence  $(2, 2, 2, 2, 3, 4, 4, 5)$  is graphic (show your work).

## Solution

$(1, 1, 3, 3)$  is an example. We see that it is non-graphic by Havel-Hakimi: the next step is  $(0, 0, 0, 2)$  and the next is  $(0, 0, -1, -1)$ .

Using Havel-Hakimi we go

$$\begin{aligned} & (2, 2, 2, 2, 3, 4, 4, 5) \\ \rightarrow & (2, 2, 1, 1, 2, 3, 3, 0) \quad \rightarrow (0, 1, 1, 2, 2, 2, 3, 3) \\ \rightarrow & (0, 1, 1, 2, 1, 1, 2, 0) \quad \rightarrow (0, 0, 1, 1, 1, 1, 2, 2) \\ \rightarrow & (0, 0, 1, 1, 1, 0, 1, 0) \quad \rightarrow (0, 0, 0, 0, 1, 1, 1, 1) \\ \rightarrow & (0, 0, 0, 0, 1, 1, 0, 0) \quad \rightarrow (0, 0, 0, 0, 0, 0, 1, 1) \\ \rightarrow & (0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

4. (a) What is the maximum number of edges in an graph with  $n$  vertices and 3 components?

- (b) What is the maximum number of triangles (copies of  $K_3$ ) in a graph with  $n$  vertices?
- (c) What is the maximum number of edges in a graph on  $n$  vertices with no triangles?

**Solution**

- (a)  $\binom{n-2}{2}$
- (b)  $\binom{n}{3}$
- (c)  $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$

5. (a) What is an Eulerian graph?
- (b) Show that every vertex of an Eulerian graph has even degree.
- (c) Characterise all graphs that have a tour, not necessarily closed, covering all edges.

**Solution**

A graph  $G$  is Eulerian if it contains a closed walk

$$W : v_0, e_1, v_1, e_2, \dots, v_{m-1}, e_m$$

such that each edge  $e \in E(G)$  occurs exactly once.

Each vertex  $v \in V(G)$  is in two edges,  $e_{i-1}$  and  $e_i$  for each time it occurs as  $v_i$  in the walk  $W$ . (As  $G$  has no loops,  $v_i$  and  $v_{i+1}$  cannot both be  $v$ , so no edge is ever counted twice.)

6. Show that a tree with a vertex of degree  $k$  has at least  $k$  leaves.

**Solution**

We know that  $m = n - 1$  so

$$2n - 2 = \sum_{v \in V(T)} \deg(v) = \ell + \sum_{v \in V^*} \deg(v),$$

where  $\ell$  is the number of leaves of  $T$  and  $V^*$  is the set of non-leaf vertices. Solving for  $\ell$  and observing that  $V^*$  contains a vertex of degree  $k$  and at  $n - \ell - 1$  other vertices of degree at least 2, we get

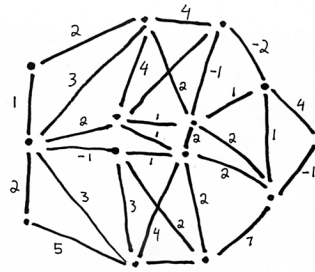
$$\ell = 2n - 2 - \sum_{v \in V^*} \deg(v) \leq 2n - 2 - 2(n - \ell - 1) - k = 2\ell - k.$$

Thus  $k \leq \ell$ , which means there are at least  $k$  leaves.

**Or, some people had a nice idea:**

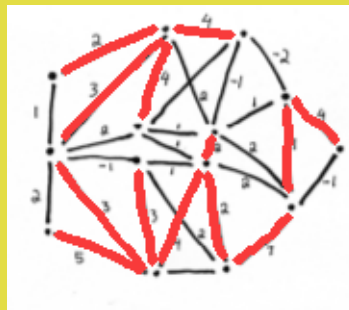
Remove the vertex  $v$  of degree  $k$  from the tree  $T$  and you are left with  $k$  subtrees. Each subtree has at least two leaves, and at most one of these leaves was adjacent to  $v$ , so is not a leaf of  $T$ . Thus at least one leaf of each of the  $k$  subtrees is a leaf of  $T$ , and so  $T$  has at least  $k$  leaves.

7. Find the weight of a maximum spanning tree of the following graph (Any



edges without weights have weight 0.)

**Solution**



$$7 + 5 + 4(4) + 3(3) + 3(2) + 1 = 44.$$

8. (a) Give Euler's formula. (Explain it: when it holds, and what your letters mean.)

- (b) Let  $G$  be a planar graph that contains no triangles. Where  $G$  has  $n$  vertices and  $m$  edges, show that

$$m \leq 2n - 4.$$

**Solution**

For a connected planar drawing of a graph  $G$  with  $n$  vertices,  $m$  edges, and  $f$  faces, Euler's formula says that  $n - m + f = 2$ .

For a planar drawing of  $G$ , we count the number  $X$  of pairs  $(F, e)$ , where  $F$  is a face of  $G$  and  $e$  is an edge on its boundary. Counting, over the faces, the number of edges bounding the face, we get, where  $f$  is the number of faces,

$$X \geq 4f.$$

Counting, over all edges, the number of faces the edge bounds, we get

$$X = 2m.$$

So  $2m \geq 4f$ , and so  $m/2 \geq f$ .

Plugging into Euler's formula  $n - m + f = 2$  we get  $n - m/2 \geq 2$ , so  $m \leq 2n - 4$ .

9. Show that if  $a \in \mathbb{Z}_m$  has a multiplicative inverse, then it has only one.

**Solution**

Let  $a$  have a multiplicative inverse  $b$ . So  $ba = 1 = ab$ . Suppose  $b'$  is a (possibly other) multiplicative inverse. Then

$$b' = b'(1) = b'(ab) = (b'a)b = (1)b = b.$$

So the inverse is unique.

10. Let  $P = (X, \mathcal{L})$  be a projective plane in which every line contains  $n$  points.
- (a) Show that every point is in  $n$  lines.
- (b) How many points does  $X$  have? (Show this; don't use the formula.)

**Solution**

Let  $x$  be a point, and let  $L$  be a line not containing it. For each of the  $n$  points  $y$  in  $L$  there is a line through  $x$  and  $y$ , and these must be distinct for distinct  $y$  (or else there are two lines containing  $y$  and  $y'$ ). So  $x$  is in at least  $n$  lines. As each line containing  $x$  intersects  $L$ ,  $x$  is in exactly  $n$  lines.

$|X| = n^2 - n + 1$ . Fix a point  $x$ . It is in  $n$  lines. Each of these lines contains  $n$  points ( $x$  and  $n - 1$  others) so there are at least  $1 + n(n - 1) = n^2 - n + 1$  points. There cannot be more, as if there were they would not be in a line with  $x$ .