

Discrete Math Midterm Solutions

1. (a) Complete the definition: For a set X , the powerset 2^X is
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(b) Show that if X and Y are sets such that $2^X = 2^Y$, then $X = Y$.

(c) (Maybe hard.) Let X and Y be sets with $|X| = |Y| = n$. Show that if $|2^X \cap 2^Y| > 2^{n-1}$ then $2^X = 2^Y$.

Solution

For a set X , the powerset 2^X is the set of all subsets of X .

Observe, at first, that for any set X , $X = \cup_{S \in 2^X} S$. Indeed, as $X \in 2^X$, we clearly have $X \in \cup_{S \in 2^X} S$. On the other hand, as for any $S \in 2^X$ we have $S \subset X$, we clearly have that $\cup_{S \in 2^X} S \in X$.

Let $I = 2^X \cap 2^Y$ we show that $X = \cup_{S \in I} S$. Indeed, for any $x \in X$, x is in 2^{n-1} sets in 2^X , so as $|I| > 2^{n-1}$, x is in some set in I . So $x \in \cup_{S \in I} S$. This was for any $x \in X$, so $X \subset \cup_{S \in I} S$. On the other hand $\cup_{S \in I} S \subset \cup_{S \in 2^X} S \subset X$, so $X = \cup_{S \in I} S$. But the same argument shows that $Y = \cup_{S \in I} S$ so $X = Y$.

2. (a) Complete the definition: For relations R and S on a set X , the composition of R and S is

$$R \circ S = \underline{\hspace{10em}}$$

(b) For relations R, S and T show that $(R \circ S) \circ T = R \circ (S \circ T)$.

Solution

For sets R and S , the composition of R and S is

$$R \circ S = \{(x, z) \mid \exists y \in X, xRy, ySz\}.$$

Assume that $(w, z) \in (R \circ S) \circ T$. This can be written as $w((R \circ S) \circ T)z$.
And

$$\begin{aligned} w((R \circ S) \circ T)z &\iff \exists y : w(R \circ S)y, yTz \\ &\iff \exists x, y : wRx, xSy, yTz \\ &\iff \exists x : wRx, x(S \circ T)z \\ &\iff w(R \circ (S \circ T))z \end{aligned}$$

which means that $(w, z) \in R \circ (S \circ T)$.

3. (a) Complete: For a poset $P = (X, \leq)$, an element x is an immediate predecessor of an element y , written $x \triangleleft y$ if

- (b) For the usual ordering \leq on \mathbb{R} show that the element 1 has no immediate predecessor.

Solution

- a) $x < y$ and there is no t such that $x < t < y$.
 b) Assume, towards contradiction, that $x \triangleleft 1$. Then by definition, $x < 1$. But then where $y = \frac{x+1}{2}$ we have $x < y < 1$, which is a contradiction.

4. (a) Draw the Hasse Diagram of the poset \mathcal{B}_3 .
 (b) Draw the Hasse Diagram of a linear extension of \mathcal{B}_3 .
 (c) How many linear extensions are there of \mathcal{B}_3 .

Solution

It looks like the figure on page 48 of the text. Only the labels are different.

There are $36 + 12 = 48$ linear extensions of \mathcal{B}_3 .

Here is why: Let $f : \mathcal{B} \rightarrow (\{0, 1, \dots, 7\}, \leq)$ be a linear extension of \mathcal{B}_3 . Then clearly $f(000) = 0$ and $f(111) = 7$. Any choice of 1, 2, 3 for $f(001)$, $f(010)$ and $f(100)$ (along with any choice of 4, 5, 6 for $f(011)$, $f(101)$ and $f(110)$) is allowed. There are 36 such choices. If $f(001) > 3$ then we must have: $f(001) = 4$, $f(110) = 3$, and $f(011) > 4$, $f(101) > 4$, $f(100) < 3$ and $f(010) < 3$. There are 4 ways to choose these values. Similarly for each of $f(100) > 3$ and $f(010) > 3$ we have 4 linear extensions.

5. How many injective functions are there from the set $[m]$ to the set $[n]$?

Solution

There are n possible targets for the first elements of $[m]$, then $n - 1$ for the next, etc: $\frac{n!}{(n-m)!}$.

6. (a) How many cycles are there in the permutation $\sigma = (4, 1, 2, 3, 8, 7, 6, 9, 5)$ of $[9]$.

- (b) (Little bit hard.) How many permutations of $[n] = \{1, 2, \dots, n\}$ have exactly two cycles.

Solution

Writing σ in cycle notation as $((1, 4, 3, 2), (5, 8, 9), (6, 7))$ we see there are 3 cycles.

When the cycles are of size a and $b = n - a$ there are $\binom{n}{a}$ ways to choose which items go in which cycle, and then $(a - 1)!(b - 1)!$ ways to order the terms within the cycles. Overall there are

$$\frac{1}{2} \sum_{i=1}^{n-1} (a - 1)!(b - 1)! \binom{n}{a} = \frac{1}{2} \sum_{i=1}^{n-1} \frac{n!}{a(n - a)}$$

such permutations. We divide by 2, because the case $a = i$ and the case $a = n - i$ are counting the same permutations.

7. How many ways can we distribute 20 indistinguishable balls among 4 distinguishable bins B_1, B_2, B_3, B_4 such that bin B_i gets at least i balls?

Solution

This is the number of positive integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 20$$

where $x_i \geq i$. Setting $y_i = x_i - i$, this is the number of non-negative integer solutions to

$$y_1 + y_2 + y_3 + y_4 = 10.$$

We know this is $\binom{10+3}{3} = \binom{13}{3}$.

8. (a) What is a fixed point of a permutation σ of $[n]$?
 (b) What is a derangement of $[n]$?
 (c) How many derangements are there of $[10]$?

Solution

A fixed point is a number i such that $\sigma(i) = i$.

A derangement σ of $[n]$ is a permutation with no fixed points.

For $i = 1, \dots, 10$, let A_i be the set of permutations of $[10]$ that fix the element i . So for any $I \in \binom{[10]}{i}$ we have that $|\cap_{i \in I} A_i| = (10 - i)!$, and we want to count $|\overline{\cap_{i=1}^{10} A_i}| = |\cap_{i=1}^{10} \bar{A}_i|$. We use the dual form of Inclusion-Exclusion:

$$\begin{aligned} |\cap_{i=1}^{10} \bar{A}_i| &= \sum_{k=0}^{10} (-1)^k \sum_{I \in \binom{[10]}{k}} |\cap_{i \in I} A_i| \\ &= \sum_{k=0}^{10} (-1)^k \binom{10}{k} (10 - k)! = \sum_{k=0}^{10} (-1)^k \frac{10!}{k!} \\ &= 10! \sum_{k=0}^{10} (-1)^k \frac{1}{k!} \end{aligned}$$