Linear Algebra
2 2019 Midterm Solutions

1. Diagonalise the matrix $A = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 3 \end{smallmatrix}\right]$.

Solution

Using the standard characteristic polynomial method you can find the eigen values $\lambda_1 = \frac{5+\sqrt{13}}{2}$ and $\lambda_2 = \frac{5+\sqrt{13}}{2}$ and the correspoding eigenvectors $x_1 = (\frac{-1+\sqrt{13}}{2}, 1)$ and $x_2 = (\frac{-1-\sqrt{13}}{2}, 1)$. (These can look very different under some scalar multiplication.) With this we diagonalise

$$\begin{aligned} A &= \begin{bmatrix} \frac{-1+\sqrt{13}}{2} & \frac{-1-\sqrt{13}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5+\sqrt{13}}{2} & \\ & \frac{5+\sqrt{13}}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{13}}{2} & \frac{-1-\sqrt{13}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{-1+\sqrt{13}}{2} & \frac{-1-\sqrt{13}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5+\sqrt{13}}{2} & \\ & \frac{5+\sqrt{13}}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \frac{1+\sqrt{13}}{2} & \frac{-1+\sqrt{13}}{2} \end{bmatrix} \frac{1}{\sqrt{13}} \end{aligned}$$

2. Choose a third vector that is orthogonal to both $(1, 1, 1)$ and $(1, -1, 0)$, and find a matrix whose eigenvectors are these three vectors, and whose eigenvalues are $1, 3, -3$.

Solution

A third orthogonal eigenvector (a, b, c) has $0 = (a, b, c)^T(1, -1, 0) = a - b$ so $a = b$ and dotting with $(1, 1, 1)$ gives that $c = -2a$. So we take $(1, 1, -2)$. Then the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -3 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -3 & \\ & & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 3 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \frac{1}{6} = \frac{1}{6} \begin{bmatrix} -4 & 14 & -6 \\ 14 & -4 & -6 \\ 8 & 8 & -12 \end{bmatrix}$$

has the eigenvalues and eigenvectors we need. (Dropping the $\frac{1}{6}$ doesn't change this.)

3. We play a game. On the 0^{th} turn a player is put in state 1 with probability $p_1 = .5$ in state 2 with probability $p_2 = .3$ and in state 3 with probablilty $p_3 = .2$. For each subsequent turn the player moves from state i to state j with probability t_{ij}, where

$$T = [t_{ij}] = \begin{bmatrix} 1/2 & 1/2 & 2/3 \\ 1/4 & 1/2 & 1/3 \\ 1/4 & 0 & 0 \end{bmatrix}.$$

You may use the approximate diagonalisation $T = M\Lambda M^{-1}$ where

$$M = \begin{bmatrix} 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{5} & \frac{-11}{5} \\ \frac{1}{4} & \frac{-6}{5} & \frac{6}{5} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & & \\ & -1/5 & \\ & & 1/5 \end{bmatrix}$$

(a) What is the probability that a player in state 2 stays in state 2 on a given turn.

(b) What is the probability of a player being in state 2 after the first turn?

(c) What is the stable state of the game?

(d) What is the probability (approximately) of a player being in state 2 after the 10^{th} turn?

(e) What is the probability of player who is in state 1 after the 2^{nd} turn ends up in state 2 after the 10^{th} turn?

Solution

(a) It is $t_{22} = 1/2$.

(b) The probablility is the second entry of Tp so is $(1/4, 1/2, 1/3)^T(.5, .3, .2) = 41/120 \approx .341$.

(c) The eigenvector with eigenvalue 1 is $(1, 2/3, 1/4)$, normalising this so that its entries sum to 1, we get that the stable state is $(12/23, 8/23, 3/23)$.

(d) This is the second entry of $T^{10}p = M\Lambda^{10}M^{-1}p$. We can compute this, but observe that $\Lambda^{10} \approx \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, so we are approximately in the stable state. of $(12/23, 8/23, 3/23)$ (which is $(1, 2/3, 1/4)$ normalised to sum to 1). Thus the probability is $8/23 \approx .347$.

(e) We need the second entry of $T^{10-2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Again $\Lambda^8 \approx \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$, and so this is approximately the stable state. So again, about .347.

4. Let $G_{k+3} = 2G_{k+2} - G_{k+1} + G_k$.

(a) Where A is the transition matrix such that for $x_k^T = [G_{k+2}, G_{k+1}, G_k]$, $x_{k+1} = Ax_k$ find an expression for A^n.

(b) If $G_0 = 1$, $G_1 = 1$ and $G_2 = 2$, find a closed expression for G_n, (that is, solve the difference equation.)

Solution

A is $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and to find A^n we must diagonalise A. Using a computer, (you were allowed to use computers!) we get $A \approx L\Lambda L^{-1}$ where

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0.56 & 0.21 + 1.30i & 0.21 - 1.30i \\ 0.32 & -1.66 + 0.56i & -1.66 - 0.56i \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1.75 & & \\ & 0.12 - 0.74i & \\ & & 0.12 + 0.74i \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} 0.95 & -0.23 & 0.54 \\ 0.02 + 0.21i & 0.11 - 0.41i & -0.27 + 0.07i \\ 0.02 - 0.21i & 0.11 + 0.41i & -0.27 - 0.07i \end{bmatrix}$$

and so $A^n \approx L \begin{bmatrix} 1.75^n & & \\ & (0.12 - 0.74i)^n & \\ & & (0.12 + 0.74i)^n \end{bmatrix} L^{-1}$.

Now G_k is approximately the bottom row of

$$\begin{aligned} LA^nL^{-1}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} &= (1.75)^n(.95, -.23, .54) \cdot (2, 1, 1) \begin{bmatrix} 1 \\ .56 \\ .32 \end{bmatrix} \\ &+ (0.12 - 0.74i)^n(0.02 + 0.21i, 0.11 - 0.41i, -0.27 + 0.07i) \cdot (2, 1, 1) \begin{bmatrix} 1 \\ 0.21 + 1.30i \\ -1.66 + 0.56i \end{bmatrix} \\ &+ (0.12 + 0.74i)^n(0.02 - 0.21i, 0.11 + 0.41i, -0.27 - 0.07i) \cdot (2, 1, 1) \begin{bmatrix} 1 \\ 0.21 - 1.30i \\ -1.66 - 0.56i \end{bmatrix} \\ &\approx (1.75)^n 2.21 \begin{bmatrix} 1 \\ .56 \\ .32 \end{bmatrix} + (0.12 - 0.74i)^n(-.26 + .08i) \begin{bmatrix} 1 \\ 0.21 + 1.30i \\ -1.66 + 0.56i \end{bmatrix} + (0.12 + 0.74i)^n(-.26 - .08i) \begin{bmatrix} 1 \\ 0.21 - 1.30i \\ -1.66 - 0.56i \end{bmatrix} \end{aligned}$$

So is

$$G_k \approx .71(1.75)^n - .38(0.12 - 0.74i)^n - .38(0.12 + 0.74i)^n \approx .53(1.75)^n$$

where the second approximation is for big n.

5. For each of the following statements, prove it if it is true, or give a counterexample if it is false.

   (a) If A is Hermitian then $A + iI$ is invertible (where $i = \sqrt{-1}$).

   (b) If Q is orthogonal, then $Q + \frac{1}{2}I$ is invertible.

   (c) If A is real then $A + iI$ is invertible.

Solution

(a) True. If H is hermitian, then all its eigenvalues are real. So the imaginary part of the eigenvalues of $A + iI$ are all i. In particular they are all non-zero, so $A + iI$ is invertible.

(b) True. Any vector v in the nullspace of $Q + \frac{1}{2}I$ would have $Qv = -\frac{1}{2}v$, but transformation by orthogonal matrixes preserves the length of vectors. So only the zero vector can be in the nullspace, meaning that $Q + \frac{1}{2}I$ is invertible.

(c) False, Any real matrix with an eigenvalue of $-i$ is a counterexample, as then $A + iI$ has a zero eigenvalue. The matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has charpoly $\lambda^2 + 1$ having roots $\pm i$, so is such a matrix.

6. Prove that if a matrix M has n linearly independent eigenvectors, then so does A^T.

Solution

If M has n linearly independent eigenvectors, then the algebraic and geometric multplicity of each eigenvalue are the same, so A is diagonalisable: $A = MDM^{-1}$, and so $A^T = (MDM^{-1})^T = (M^{-1})^T D^T M^T = (M^T)^{-1} DM^T$ is also diagonalisable. Thus its has n linearly independent eigenvectors.

7. Let $f(x, y, z) = x^2 + 2y^2 + 11z^2 - 2xy - 2xz - 4zy - 2x + y - z$, and $v = (x, y, z)$. Find the symmetric matrix A and vector b such that $f(x, y, z) = \frac{1}{2}v^T Av - b^T v$. Show that A is positive definite, and find the minimum and minimum value of $f(x, y, z)$.

Solution

We have that $A = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 4 & -4 \\ -2 & -4 & 22 \end{bmatrix}$ and $b = (-2, 1, -1)$. To see that A is positive semidefinite it is enough to eliminate it by Gaussian eliminate to $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{bmatrix}$ which has positive pivots. As it is positive semidefinite $f(x, y, z)$ has a unique critical point which is a minimum, and it occurs where

$$0 = f_x(x, y, z) = 2x - 2y - 2z - 2$$
$$0 = f_y(x, y, z) = -2x + 4y - 4z + 1$$
$$0 = f_z(x, y, z) = -2x - 4y + 22z - 1$$

which we recognise as the solution of $0 = Av + b$. Thus the minimum is at

$$v = A^{-1}(-b) = \frac{1}{2}\begin{bmatrix} 18 & 13 & 4 \\ 13 & 10 & 3 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 27/2 \\ 19/2 \\ 3 \end{bmatrix}.$$

Plugging this into the polynomial, we get that the mininum value is

$$-\frac{1}{2}b^T v = (1, -1/2, 1/2)^T (27/2, 19/2, 3) = 47/4.$$