Linear Algebra 2 2019 Midterm Test

- Hand you solutions in in class on Oct 30 (Wednesday), or at my office before that.
- Write neatly, especially your name and student number.
- You are allowed to use computers to compute and matrix products, inverses, eigenvalues, or diagonalisations. Except in the first question– I want you to do that by hand. If you use a computer, be clear about what you have done. I don't want just the anwers. (You can use www.wolframalpha.com or www.cocalc.com to compute these things.)
- Clearly $\boxed{\text{box}}$ your final answer if there is one.
- There are 7 questions worth 5 points each for a total of 35.

1. Diagonalise the matrix $A = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 3 \end{smallmatrix}\right]$.
2. Choose a third vector that is orthogonal to both $(1, 1, 1)$ and $(1, -1, 0)$, and find a matrix whose eigenvectors are these three vectors, and whose eigenvalues are $1, 3, -3$.
3. We play a game. On the 0^{th} turn a player is put in state 1 with probability $p_1 = .5$ in state 2 with probability $p_2 = .3$ and in state 3 with probablilty $p_3 = .2$. For each subsequent turn the player moves from state i to state j with probability t_{ij}, where

$$T = [t_{ij}] = \begin{bmatrix} 1/2 & 1/2 & 2/3 \\ 1/4 & 1/2 & 1/3 \\ 1/4 & 0 & 0 \end{bmatrix}.$$

   You may use the approximate diagonalisation $T = M\Lambda M^{-1}$ where

$$M = \begin{bmatrix} 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{5} & \frac{-11}{5} \\ \frac{1}{4} & \frac{-6}{5} & \frac{6}{5} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & & \\ & -1/5 & \\ & & 1/5 \end{bmatrix}$$

   (a) What is the probability that a player in state 2 stays in state 2 on a given turn.

   (b) What is the probability of a player being in state 2 after the first turn?

   (c) What is the stable state of the game?

   (d) What is the probability (approximately) of a player being in state 2 after the 10^{th} turn?

   (e) What is the probability of player who is in state 1 after the 2^{nd} turn ends up in state 2 after the 10^{th} turn?

4. Let $G_{k+3} = 2G_{k+2} - G_{k+1} + G_k$.

   (a) Where A is the transition matrix such that for $x_k^T = [G_{k+2}, G_{k+1}, G_k]$, $x_{k+1} = Ax_k$ find an expression for A^n.

   (b) If $G_0 = 1$, $G_1 = 1$ and $G_1 = 2$, find a closed expression for G_n, (that is, solve the difference equation.)

5. For each of the following statements, prove it if it is true, or give a counterexample if it is false.

   (a) If A is Hermitian then $A + iI$ is invertible (where $i = \sqrt{-1}$).

   (b) If Q is orthogonal, then $Q + \frac{1}{2}I$ is invertible.

   (c) If A is real then $A + iI$ is invertible.

6. Prove that if a matrix M has n linearly independent eigenvectors, then so does A^T.

7. Let $f(x, y, z) = x^2 + 2y^2 + 11z^2 - 2xy - 2xz - 4zy - 2x + y - z$, and $v = (x, y, z)$. Find the symmetric matrix A and vector b such that $f(x, y, z) = \frac{1}{2}v^T Av - b^T v$. Show that A is positive definite, and find the minimum and minimum value of $f(x, y, z)$.

Question	Out of	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Total	35