Linear Algebra 2019 Final Solutions

1. Answer the following with True (T) or False (F).

   (a) If $x \in W$ then $\text{proj}_W\, x = 0$.

   (b) If $S = T^\perp$ then $T = S^\perp$.

   (c) The projection matrix for the space generated by $(1, 0, 0)$ and $(1, 1, 0)$, is invertible.

   (d) If $\det(M) = 4$ then $\det(M - I) = 3$.

   (e) The cofactor matrix of an invertible matrix M has determinant 1.

Solution

(a) F $\text{proj}_W\, x = x$

(b) T $S = (S^\perp)^\perp$.

(c) F It has rank 2 so doesn't have full rank. .

(d) F If $M = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ the $\det(M) = 4$ but $\det(M - I) = \begin{vmatrix} 3 & 0 \\ 0 & 0 \end{vmatrix} = 0$

(e) T If M is invertible, then $M^{-1} = \frac{C^T}{\det(M)}$ where C is the cofactor matrix of M. So $1 = \det(M^{-1})\det(M) = \det C^T = \det C$.

2. Find a basis for the orthogonal complement $A^\perp$ of the space A spanned by $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$.

Solution

$A^\perp$ is the nullspace of $M = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}$ which we find by solving $Mx = 0$ by elimination:

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

The nullspace is $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, which has a basis $\boxed{\{ (0,-1,1,0), (-5,1,0,1) \}}$.

3. Find the matrix that projects every point in the plane $\mathbb{R}^2$ onto the line $x - 3y = 0$. What is the projection p of the point $(1, 2)$ onto this line?

Solution

This line is spanned by the vector $a = (3, 1)$. The projection matrix for this is

$$P = \frac{aa^T}{a^Ta} = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}.$$

The projection of $(1, 2)$ is $\frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 5 \end{bmatrix}$.

4. Find a least squares approximation $\hat{x}$ to a solution of $Ax = b$ where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}.$$

Solution

We compute $\hat{x} = (A^TA)^{-1}A^Tb$.

$$A^TA = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}, (A^TA)^{-1} = \frac{1}{56}\begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix}.$$

And $A^tb = (-24, -2)$ so then

$$(A^TA)^{-1}A^Tb = \frac{1}{14}\begin{bmatrix} 5 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} -12 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

5. If P is the projection onto the column-space of A, what is the projection onto the left nullspace? (Why?)

Solution

The left-nullspace is the orthogonal complement of the columnspace, so to project a vector onto it, we simply have to remove the part that projects onto the columnspace. Thus $I - P$ is the projection onto the left nullspace.

6. Let M be an $n \times n$ matrix whose columns are pairwise orthonormal. If every entry of M is $1/4$ or $-1/4$, then what is n? (Explain why.)

Solution

16. The inner product of any row with itself is $1/16 + 1/16 + \cdots + 1/16$ with n summands, and must equal 1 for the matrix to be orthogonal.

7. Let $a = (2, 2, 2)$, $b = (1, -1, 0)$ and $c = (2, 0, 1)$.

(a) Show that a and b are orthogonal.

(b) Show that c is in the space generated by a and b.

(c) Find an orthogonal basis, containing a and b, for the space generated by a, b, and $d = (1, 2, 3)$.

Solution

(a) $(2, 2, 2)(1, -1, 0)^T = 2 - 2 + 0 = 0$

(b) $.5a + b = (1, 1, 1) + (1, -1, 0) = (2, 0, 1) = c$.

(c) We remove from d the projections onto a (or $a/2$ to simplify calculation) and b:

$$\begin{aligned} d' &= d - \text{proj}_a(d) - \text{proj}_b(d) \\ &= (1,2,3) - \frac{(1,1,1)\cdot(1,2,3)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(1,-1,0)\cdot(1,2,3)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0) \\ &= (1,2,3) - \frac{6}{3}(1,1,1) - \frac{-1}{2}(1,-1,0) = (-1/2,-1/2,1). \end{aligned}$$

So $\{a, b, (1, 1, -2)\}$ is a required basis.

Alternately, you could observe that d is not in the span of a and b, so the space we want a basis of is all of $\mathbb{R}^3$. It is then enough to find any vector orthogonal to a and b as the third vector of the basis. Such a vector will be a scale of $(1, 1, -2)$.

8. If a 4×4 matrix M has $\det(M) = 1/3$ what are

(a) $\det(2M)$

(b) $\det(M^{-1})$

(c) $\det(-M)$

(d) $\det(M^T M)$

Solution

(a) 16/3

(b) 3

(c) 1/3

(d) 1/9

9. (a) What does it mean for a matrix Q to be orthogonal?

(b) Show that an orthogonal matrix Q has determinant -1 or 1.

(c) Describe the parallelopiped whose edges are the column vectors of an orthogonal matrix Q.

Solution

(a) Its columns are pairwise ortha**normal**. (Pairwise orthagonal is not enough!) Equivalenly $Q^TQ = I$.

(b) As $1 = |I| = |Q^TQ| = |Q||Q|$ we have that $|Q|$ is a square root of 1, so is ± 1.

(c) It is a unit n-dimensional cube: sides are orthoganal and have unit length.

10. Find the determinants of

(a) $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 1 & 2 & 2 \\ 4 & 4 & 8 & 8 \end{bmatrix}$

Solution

(a) $2 - (-1) = 3$

(b) Expand along second row: $2(2-4) - 2(2-3) = -2$.

(c) 16 Two row switches make it upper triangular.

11. Use Cramer's rule to find x_2 where $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Solution

$$x_2 = \frac{\begin{vmatrix} 2 & 1 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & 7 \\ 0 & 1 & 4 \\ 0 & 2 & 4 \end{vmatrix}} = \frac{16}{-8} = -2.$$

12. Find the area of the triangle with vertices (corners) at the points $(2,3)$, $(4,1)$ and $(6,4)$.

Solution

This is half of the parallelopiped defined by the vectors $v = (2, -2)$ and $u = (4, 1)$, so has area

$$1/2 \left| \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} \right| = 5.$$

Many people used the trick described in Section 4.4 question #2(b) and computed $1/2 \left| \begin{smallmatrix} 2 & 3 & 1 \\ 4 & 1 & 1 \\ 6 & 4 & 1 \end{smallmatrix} \right|$ to get the same answer.