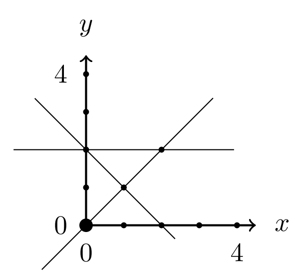
Linear Algebra 2019 Midterm Solutions

1. Answer 'T(rue)' or 'F(alse)'.
   (a) If $M^2 + M = I$ then $M^{-1} = M + I$.
   (b) If the diagonal entries of a matrix M are 0, then M is singular.
   (c) A matrix M is invertible if and only if it has 3 pivots.
   (d) The matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has rank 2.
   (e) There are n different $n \times n$ permutation matrices.
   (f) The LU decomposition of a matrix is unique if it exists.
   (g) Any three independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$.
   (h) Given a matrix A, if we reduce the augmented matrix $[A \mid I]$ by Gaussian Elimination to $[A' \mid B]$, then $BA = A'$.

Solution

	T or F	Comment/Reason
a	T	$1 = M^2 + M \Rightarrow I = M(M + I)$
b	F	The permutation matrix $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ is invertible.
c	F	Only true for 3×3 matrices.
d	T	Two pivots
e	F	There are $n!$ of them.
f	F	LDU is.
g	T	
h	T	

2. Given a system of three equations in two variables (x and y), where the equations define the shown lines, find all solutions to the system.



Solution

There are no solutions. There are no points in which all three lines intersect.

3. Find a, b and c where

$$\begin{bmatrix} a & 2 \\ 1 & 1 \\ b & 4 \end{bmatrix} \begin{bmatrix} c \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}.$$

Solution

From the second row we get that $c = 2$. So $a = -1$ and $b = -3/2$.

4. (a) Write out the 3×3 row operation matrix E_1 that adds three copies of row 1 to row 2.

   (b) Write out the 3×3 row operation matrix E_2 that switches row 1 with row 3.

   (c) Compute E_1^n, and E_2^n.

Solution

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. E_1^n adds three copies of row 1 to row 2 n times, so add $3n$ copies of row 1 to row 3: $E_1^n = \begin{bmatrix} 1 & 0 & 0 \\ 3n & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. E_2^n switches these two rows n times, so is I if n is even E_2 if n is odd.

5. Find an LDU decomposition of the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 7 & 2 \end{bmatrix}.$$

Solution

Eliminating the augmented matrix give

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -5 & -10 & 3 & 1 \end{array}\right].$$

By inspection, we get the inverse of $L^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -10 & 3 & 1 \end{bmatrix}$: What do we need to multiply by to get the identity? The first row should be $(1,0,0)$. For the second, we need 2 times the first row of L^{-1} minus one times the second, so the second row is $(2,-1,0)$. Similar inspection of the third row gives $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$. So

$$\begin{aligned} B &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

6. Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -2 & 1 & 1 & -3 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Solution

Sorry! This was a long tedious question. I gave you full marks if you started it properly. Reduce the augmented matrix:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & -3 & 0 & 1 & 0 & 0 \\ -1 & 3 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 23/3 & 4 & -4/3 & 17/3 \\ 0 & 1 & 0 & 0 & 11/3 & 2 & -1/3 & 8/3 \\ 0 & 0 & 1 & 0 & 5/3 & 1 & -1/3 & 5/3 \\ 0 & 0 & 0 & 1 & -10/3 & -2 & 2/3 & -7/3 \end{array}\right]$$

So $1/3\begin{bmatrix} 23 & 12 & -4 & 17 \\ 11 & 6 & -1 & 8 \\ 5 & 3 & -1 & 5 \\ -10 & -6 & 2 & -7 \end{bmatrix}$.

7. Which of the following subsets of $\mathbb{R}^3$ are subspaces:

(a) The set of vectors (b_1, b_2, b_3) with $b_1 = 1$.

(b) The set of vectors (b_1, b_2, b_3) with $b_1 = 0$.

(c) The set of vectors (b_1, b_2, b_3) where b_1 is integer.

(d) The set of vectors (b_1, b_2, b_3) with $b_1 b_2 = 0$.

(e) The set of linear combinations of the vectors $(1, 2, 6)$ and $(3, 2, 1)$.

(f) The set of vectors (b_1, b_2, b_3) with $b_1 + b_2 - 2b_3 = 3$.

Solution

These are: (b), (e).
(a) and (f) don't contain $(0, 0, 0)$ so are not closed under scalar multiplicaton.
(c) is not closed under scalar multiplication (by say $1/2$).
(d) is the union of two planes, (not the intersection): $(1, 0, 0)$ and $(0, 1, 1)$ are in the set, but their sum is not.

8. Construct a 3×3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$.

Solution

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The third column must not be independent of the other two.

9. Where A is the matrix in question 6, solve $A[w, x, y, z]^T = [2, 3, 4, 5]^T$.

Solution

Sorry about this too. Because question 6 6 was so nasty, no-one could really do this. So I made it out of 0.

$$A^{-1}[2, 3, 4, 5]^T = 1/3 \begin{bmatrix} 23 & 12 & -4 & 17 \\ 11 & 6 & -1 & 8 \\ 5 & 3 & -1 & 5 \\ -10 & -6 & 2 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = 1/3 \begin{bmatrix} 151 \\ 76 \\ 40 \\ -65 \end{bmatrix}.$$

10. The matrix equation $M\mathbf{x} = \mathbf{b}$ has solutions $(1, -1, 2)$ and $(-2, 2, -4)$. What is $\mathbf{b}$?

Solution

It is $\mathbf{0}$. As $2(1, -1, 2) + (-2, 2, -4) = (0, 0, 0)$ is also a solution to $M\mathbf{x} = \mathbf{b}$, we must have $\mathbf{b} = \mathbf{0}$.

11. How would you decide if a set of vectors of length 3 are independent? Do it for the following vectors:

$$(1, 3, 2), (2, 1, 3), \text{ and } (3, 2, 1).$$

Solution

Put them as the rows of a matrix and eliminate to determine if it has rank m (independent) or less (dependent).

$$\begin{bmatrix}1 & 3 & 2\\ 2 & 1 & 3\\ 3 & 2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 7 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 0 & 28/5\end{bmatrix}$$

Rank 3, so the three rows are independent.

(Alternately, put them as the columns of M and show that $Mx = 0$ has only the trivial solution (independent) or more (dependent).)

12. Where

$$C = \begin{bmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 1 & 0\\ 1 & 2 & 0 & 1\end{bmatrix}$$

(a) Find the solution of $Cx = 0$

(b) Find the solution of $Cx = [1, 2, 3]^T$.

(c) Give a basis of the columnspace of C.

(d) Give a basis of the rowspace of C.

(e) What is the rank of C?

Solution

$$\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 1\\ 0 & 1 & 1 & 0 & 2\\ 1 & 2 & 0 & 1 & 3\end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c}1 & 0 & -2 & 1 & -1\\ 0 & 1 & 1 & 0 & 2\\ 0 & 0 & 0 & 0 & 2\end{array}\right]$$

So

(a) $N(C) = \{z\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} + y\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}\}$

(b) No solutions: $\mathbf{0} \cdot \mathbf{x} = 2$ from the last row of the eliminated matrix, has no solution.

(c) Take the pivot columns: $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\end{bmatrix}$.

(d) The pivot rows: $(1, 0, -2, 1)$ and $(0, 1, 1, 0)$.

(e) Two!

13. (a) What matrix transforms $(1, 0, 0)$ to $(2, 3, 4)$, $(0, 1, 0)$ to $(1, -1, 2)$ and $(0, 0, 1)$ to $(1, 1, 1)$?

(b) What does this same matrix transform $(1, -1, 2)$ to?

Solution

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and (b) $\begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

14. Which of these transformation **are** linear? Where $v = (v_1, v_2)$,

   (a) $T(v) = (v_2, v_2)$.
   (b) $T(v) = (1, v_2)$.
   (c) $T(v) = (-v_2, v_1 + v_2)$.
   (d) $T(v) = (v_1^2, -(v_2^2))$.

Solution

(a) and (c) are, they are multplication by the matrices $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ respectively.

(b) isn't: $T(1, 1) = (1, 1) \neq (1, 1) + (1, 0) = T(0, 1) + T(1, 0)$.

(d) isn't: $T(2, 0) = (4, 0) \neq (1, 0) + (1, 0) = T(1, 0) + T(1, 0)$.

15. Let A and B be the matrices such that

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

What are the products AB and BA.

Solution

$AB : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4) \mapsto (0, x_2, x_3, x_4)$ is the 4×4 matrix

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

while $BA : (x_1, x_2, x_3) \mapsto (0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)$ is the 3×3 identity matrix I_3.