2022 Probability and Statistics Midterm Solutions

1. Light bulbs are produced by three factories A,B, and C which produce 20%, 30%, and 50% of the lightbulbs respectively. The lightbulbs are defective with probabilities 1%, 2% and 3% respectively.

   What is the probability that a randomly chosen defective lightbulb is from factory B?

Solution

Using Bayes formula: factory B contributes $.3*.02 = .006$ of the total defective proportion $.2*.01+.3*.02+.5*.03 = .002+.006+.015 = .023$ of lightbulbs. So the probability that a defective bulb is from factory B is $.006/.023 = 6/23$.

2. Let C_1 and C_2 be mutually independent events with probabilities $1/2$ and $1/5$ respectively. What is $P(C_1 \cap C_2^c)$?

Solution

As C_1 is independent of C_2 it is also independent of C_2^c so this probability is $1/2*(1-1/5) = 1/2*4/5 = 2/5$.

3. (a) Let X have pmf $p_X(x) = (1/2)^{-x}$ for $x \in \{-1,-2,\dots\}$. Find the pmf of $Y = X^2$.

   (b) Let X have pdf $f_x = 1/2$ on $[0,2]$. Find the pdf of $Y = X^2$.

Solution

This $Y = g(X) = X^2$ is a one-to-one function on the universe of X (for both parts) its inverse is $g^{-1}(y) = -\sqrt{y}$ for part (i) and $\sqrt{y}$ for part (ii).

So for part (i) we have $p_Y(y) = p_X(-\sqrt{y}) = (1/2)^{\sqrt{y}}$ for Y a positive square.

For part (ii), where the Jacobian is $J = \frac{\mathrm{d}}{\mathrm{d}y}x = \frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = 1/2\sqrt{y}$ we have $f_Y(y) = f_x(\sqrt{y})|J| = 1/4\sqrt{y}$ on $y \in (0,4]$.

4. Find the mean and variance of the distribution with cdf

$$F = \begin{cases} x/8 & \text{if } 0 \le x < 2 \\ x^2/16 & \text{if } 2 \le x < 4 \end{cases}$$

   on the support $[0,4]$.

Solution

Differentiating, the pdf $f(x)$ is $1/8$ on $[0,2)$ and $x/8$ on $[2,4]$.
The mean is

$$E(X) = \int_0^2 \frac{x}{8} dx + \int_2^4 \frac{x^2}{8} dx = \frac{1}{8}(2 + \frac{64-16}{3}) = \frac{9}{4}$$

the second moment is

$$E(X^2) = \int_0^2 \frac{x^2}{8} dx + \int_2^4 \frac{x^3}{8} dx = \frac{1}{8}(8/3 + \frac{4^4 - 2^4}{4}) = \frac{47}{6}$$

and so the variance is

$$Var(X) = E(X^2) - E(X)^2 = \frac{47}{6} - (\frac{9}{4})^2 \approx 3.$$

5. Let X have pdf $f(x) = 3x^2$ on the region $0 < x < 1$, (and 0 elsewhere). Find the value of x such that $P(X < x) = 1/8$ and $P(X \leq x) \geq 1/8$.

Solution

This is the x such that $1/8 = \int_0^x 3t^2\, \mathrm{d}t = [t^3]_0^x = x^3$; so $x = 1/2$.

6. Let X have pdf $f(x) = 6x(1-x)$ for its support $(0,1)$. Let $\mu = E(X)$ and $\sigma^2 = \mathrm{Var}(V)$.

   (a) Bound $P(\mu - 2\sigma < X < \mu + 2\sigma)$ using Chebyshev's inequality.

   (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$ exactly.

Solution

By Chebyshev, we have

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(|X - \mu| < 2\sigma) \geq 1 - 1/2^2 = 3/4.$$

To find the value exactly we have to first find μ and σ: As $f(x)$ is symmetric around $1/2$, it is clear that $\mu = 1/2$. For σ

$$\sigma^2 = E(X^2) - \mu^2 = \int_0^1 x^2 f(x)dx - 1/2^2 = 6\int_0^1 x^3 - x^4 dx - 1/4 = 1/20$$

So $\sigma = 1/(2\sqrt{5})$.
The probability is thus

$$\begin{aligned}
6\int_{1/2-1/\sqrt{5}}^{1/2+1/\sqrt{5}} x - x^2 dx &= 6\int_{-1/\sqrt{5}}^{1/\sqrt{5}} (x + 1/2) - (x + 1/2)^2 dx \\
&= 6\int_{-1/\sqrt{5}}^{1/\sqrt{5}} 1/4 - x^2 dx \\
&= 12\int_0^{1/\sqrt{5}} 1/4 - x^2 dx \\
&= 11\sqrt{5}/25
\end{aligned}$$

7. Let (X, Y) be the random vector with joint pdf $f_{X,Y}(x, y) = 6y$ on its support $0 \leq y \leq x \leq 1$.

   (a) Find the marginal pdf $f_Y(y)$ of Y.

   (b) Find pdf $f_{X|y}(x)$ of the conditional distribution $X|y$.

   (c) Find the mean of $X|y$.

Solution

For fixed y, $f_{X,Y}(x,y)$ has support $y \le x \le 1$, so the marginal pdf of Y is

$$f_Y(y) = \int_y^1 f_{X,Y}(x,y)\,\mathrm{d}x = \int_y^1 6y\,\mathrm{d}x = 6y(1-y).$$

The conditional pdf is

$$f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{6y}{6y(1-y)} = \frac{1}{1-y}.$$

The mean of $X|y$ is

$$\mu = \int_y^1 x f_{X|y}(x)\,\mathrm{d}x = \frac{1}{1-y}\int_y^1 x\,\mathrm{d}x = \frac{1}{1-y}(\frac{1-y^2}{2} = \frac{1+y}{2}.$$

The second moment is

$$E((X|y)^2) = \int_y^1 x^2 f_{X|y}(x)\,\mathrm{d}x = \frac{1}{1-y}(\frac{1-y^3}{3}) = \frac{1+y+y^2}{3},$$

and so the variance is

$$\sigma^2 = E((X|y)^2) - \mu^2 = \frac{y^2-2y+1}{12}.$$

8. Let X and Y have the joint pdf $f(x,y) = 3x$ for $0 < x < y < 1$ (and 0 elsewhere). Show whether or not X and Y are independent.

Solution

The marginal pdf of X is $f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 3x\,dy = 3x(1) - 3x^2 = 3x(1-x)$, and of Y is $f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 3x\,dx = \frac{3}{2}y^2$. As $f \neq f_X \cdot f_Y$, the distributions are not independent. (Alternately one could observe that for fixed x the conditional pdf of Y is $f_{Y|X}(y \mid x)$ is uniform, and so $E(Y|X=x) = (x+1)/2$. This depends on X, so Y depends on X.)