2022 Probability and Statistics Practice Test Solutions

1. A bowl contains nine blue chips and one red one. You draw chips, without replacement, until you draw the red one. X is the number of chips you draw. What is the pmf p of X? What if you replace the chip removed after each draw?

Solution

Without replacement it is

$$p(x) = (\frac{9}{10} \cdot \frac{8}{9} \cdots \frac{10-x+1}{10-x+2}) \cdot \frac{1}{10-x+1} = 1/10$$

for $x = 1, 2, \ldots 10$.
With replacement it is $p(x) = (9/10)^{x-1} \cdot (1/10) = \frac{9^{x-1}}{10^x}$.

2. Let C_1, C_2, C_3 be mutually independent events. Show that $C_1 \cup C_2$ and C_3 are independent. (You may use the fact that C_i, C_j^c and C_k are also mutually independent for distinct i, j and j,)

Solution

We need to show that $P(C_1 \cup C_2 \mid C_3) = P(C_1 \cup C_2)$. Partitioning $C_1 \cup C_2$ into three disjoint sets

$$C_1 \cup C_2 = (C_1 \cap C_2^c) \cup (C_1 \cap C_2) \cup (C_1^c \cap C_2)$$

we compute

$$\begin{aligned}
P(C_1 \cup C_2 \mid C_3) &= \frac{P([C_1 \cup C_2] \cap C_3)}{P(C_3)} \\
&= \frac{P([C_1 \cap C_2^c \cap C_3] \cup [C_1 \cap C_2 \cap C_3] \cup [C_1^c \cap C_2 \cap C_3])}{P(C_3)} \\
&= \frac{P(C_1 \cap C_2^c \cap C_3) + P(C_1 \cap C_2 \cap C_3) + P(C_1^c \cap C_2 \cap C_3)}{P(C_3)} \\
&= \frac{P(C_1)P(C_2^c)P(C_3) + P(C_1)P(C_2)P(C_3) + P(C_1^c)P(C_2)P(C_3)}{P(C_3)} \\
&= P(C_1)P(C_2^c) + P(C_1)P(C_2) + P(C_1^c)P(C_2) \\
&= P([C_1 \cap C_2^c] \cup [C_1 \cap C_2] \cup [C_1^c \cap C_2]) = P(C_1 \cup C_2)
\end{aligned}$$

For the second equality we used DeMorgan, for the third and sixth equalities we used that these three sets are disjoint, and for the fourth equality we used the independence,

3. Let C_1, C_2, C_3 be mutually independent events with probabilities $1/2, 1/5$ and $1/5$ respectively. What is $P(C_1 \cup C_2 \cup C_3)$?

Solution

The probability that none of the events happens is $1/2 * 4/5 * 4/5 = 8/25$. So the probability that at least one did is $17/25$.

4. Let X have the pdf $f(x) = (x+2)/c$ for some constant c and support $-2 < x < 3$.

   (a) Find c.

   (b) Find $P(X^2 < 4)$.

Solution

To find c we solve

$$1 = 1/c \int_{-2}^{3} x + 2dx = 1/c \cdot [x^2/2 + 2x]_{-2}^{3}$$

which gives $c = 25/2$.
So

$$\begin{aligned} P((X^2 < 4) &= P(-2 < X < 2) = 2/25 \int_{-2}^{2} x + 2dx = 2/25[x^2/2 + 2x]_{-2}^{2} \\ &= 2/25[2x]_{-2}^{2} = 16/25 \end{aligned}$$

5. Find the mean and variance of the distribution with cdf

$$F = \begin{cases} x/8 & \text{if } 0 \le x < 2 \\ x^2/16 & \text{if } 2 \le x < 4 \end{cases}$$

on the support $[0, 4]$.

Solution

Differentiating, the pdf $f(x)$ is $1/8$ on $[0, 2)$ and $x/8$ on $[2, 4]$.
The mean is

$$E(X) = \int_0^2 \frac{x}{8} dx + \int_2^4 \frac{x^2}{8} dx = \frac{1}{8}(2 + \frac{64-16}{3}) = \frac{9}{4}$$

the second moment is

$$E(X^2) = \int_0^2 \frac{x^2}{8} dx + \int_2^4 \frac{x^3}{8} dx = \frac{1}{8}(8/3 + \frac{4^4 - 2^4}{4}) = \frac{47}{6}$$

and so the variance is

$$Var(X) = E(X^2) - E(X)^2 = \frac{47}{6} - (\frac{9}{4})^2 \approx 3.$$

6. Let X have pdf $f(x) = 6x(1-x)$ for its support $(0, 1)$. Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(V)$.

   (a) Bound $P(\mu - 2\sigma < X < \mu + 2\sigma)$ using Chebyshev's inequality.

   (b) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$ exactly.

Solution

By Chebyshev, we have

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(|X - \mu| < 2\sigma) \geq 1 - 1/2^2 = 3/4.$$

To find the value exactly we have to first find μ and σ: As $f(x)$ is symmetric around $1/2$, it is clear that $\mu = 1/2$. For σ

$$\sigma^2 = E(X^2) - \mu^2 = \int_0^1 x^2 f(x)dx - 1/2^2 = 6\int_0^1 x^3 - x^4 dx - 1/4 = 1/20$$

So $\sigma = 1/(2\sqrt{5})$.
The probability is thus

$$\begin{aligned} 6\int_{1/2-1/\sqrt{5}}^{1/2+1/\sqrt{5}} x - x^2 dx &= 6\int_{-1/\sqrt{5}}^{1/\sqrt{5}} (x+1/2) - (x+1/2)^2 dx \\ &= 6\int_{-1/\sqrt{5}}^{1/\sqrt{5}} 1/4 - x^2 dx \\ &= 12\int_0^{1/\sqrt{5}} 1/4 - x^2 dx \\ &= 11\sqrt{5}/25 \end{aligned}$$